MAS241 CH5: Functions of Bounded Variation

MaxLevSnail

September 18, 2022

1 Partitions

Definition 1.1. $\pi = \{x_0, x_1, ..., x_p\}$ is called a partition of $[a, b]$ if $a = x_0 < x_1 < ... < x_p = b$.

$\Delta x_j = x_j - x_{j-1}$

$\Pi[a, b]$: Collection of all partitions.

Definition 1.2. Let $\pi_1, \pi_2 \in \Pi[a, b]$ We call π_1 is finer than π_2 if $\pi_2 \subset \pi_1$ and denote $\pi_2 \preceq pi_1$. (π_2 is coarser than π_1).

Definition 1.3. $\pi_1 \vee \pi_2$: Least common refinement of π_1 and π_2.

Lemma 1.1. $\varepsilon \preceq \varepsilon$ *is a partial order in* $\Pi[a, b]$

1. $\pi \preceq \pi \ \forall \pi \in \Pi[a, b]$
2. $\pi_1 \preceq \pi_2, \pi_2 \preceq \pi_1 \Longrightarrow \pi_1 = \pi_2$
3. $\pi_1 \preceq \pi_2, \pi_2 \preceq \pi_3 \Longrightarrow \pi_1 = \pi_3$

Theorem 1.2. $\pi_1 \preceq \pi_2 \Longrightarrow \pi_1 \vee \pi_2 = \pi_2$

Definition 1.4. Gauge of partition π: $\|\pi\| = max\{\Delta x_j : j = 1..p\}$. And it is not a norm.

2 Monotone Functions

Definition 2.1. Definition about increasing/decreasing function, and it was trivial.

Theorem 2.1. $f : [a, b] \longrightarrow \mathbb{R}$ *and* f *is increasing.*
$\pi = \{x_0, ..., x_p\} \in \Pi[a, b] \Rightarrow \sum_{j=0}^{p}(f(x_j^+) - f(x_j^-)) \le f(b) - f(a)$

Theorem 2.2. $f : [a, b] \longrightarrow \mathbb{R}$ *and* f *is monotone.*
$\Rightarrow f$ *is discontinuous at most countable times.*

Proof. $A_k = \{x \in [a, b] : f(x^+) - f(x^-) > 1/k\}$
Claim: A_k is finite for all $k > 0$.
Suppose A_k has infinitely many points. Then theorem 1 fails. Hence A_k are all finite.
$\bigcup_{k=1}^{\infty} A_k \supset \{x \in [a, b]/f$ is discontinuous at $x\}$. □

Saltus functions We can describe any function to the sum of a discrete function and a continuous function.

$u(x) = f(x) - f(x^-)$ when $x \neq a$, $v(x) = f(x^+) - f(x)$ when $x \neq b$, then we can define a set S : collection of all discontinuous points of f.

Saltus function:

$$S_f(x) = f(a) + \sum_{x_j \in S \cap (a,x]} u(x_j) + \sum_{x_j \in S \cap [a,x)} v(x_j)$$

3 Function of Bounded Variation

Definition 3.1. Variation $V(f;\pi) = \sum |\Delta f_j|$
$f \in BV(a,b)$ is of bounded variation if $\exists M > 0$ such that $V(f;\pi) \leq M$ for all $\pi \in \Pi[a,b]$
$V(f;a,b) : \sup_{\pi \in \Pi[a,b]} V(f;\pi)$

Theorem 3.1. *$f \in BV(a,b) \Rightarrow f$ is bounded.*

Theorem 3.2. *If f is monotone, then $f \in BV(a,b)$*

Proof. $V(f;\pi) = \sum_{j=1}^{p} |f(x_j) - f(x_{j-1})| = \sum_{j=1}^{p} (f(x_j) - f(x_{j-1})) = f(x_p) - f(x_0) = f(b) - f(a)$ □

Theorem 3.3. *$f : [a,b] \longrightarrow \mathbb{R}$, continuous on $[a,b]$, differentiable on (a,b)*
$\|f'\|_\infty < \infty \Rightarrow f \in BV(a,b)$

Proof. $V(f;\pi) = \sum_{j=1}^{p} |f(x_j) - f(x_{j-1})| = \sum_{j=1}^{p} |f'(c_j)(x_j - x_{j-1})| \leq \sum_{j=1}^{p} \|f'\|_\infty (x_j - x_{j-1}) = \|f'\|_\infty$ □

Theorem 3.4. *$f, g \in BV(a,b) \Rightarrow f+g, f-g \in BV(a,b)$*

Theorem 3.5. *$f, g \in BV(a,b) \Rightarrow fg \in BV(a,b)$*

Theorem 3.6. *$f \in BV(a,b) \Rightarrow 1/f \in BV(a,b)$*

4 Total Variation as a Function

Theorem 4.1. *If $f \in BV(a,b) c \in (a,b)$, then $f \in BV(a,c), f \in BV(c,b)$.*
$V(f;a,b) = V(f;a,c) + V(f;c,b)$

Proof. Let $\pi = \{x_0, ..., x_p\} \in \Pi[a,c]$. Then $\overline{\pi} = \{x_0, ..., x_p, b\} \in \Pi[a,b]$
$V(f,\pi) \leq V(f,\overline{\pi}) < M$. Hence $f \in BV(a,c), f \in BV(c,b)$ □

Definition 4.1. $f \in BV(a,b)$. The variation of f is a function, $V_f : [a,b] \longrightarrow \mathbb{R}$ given by $V_f(x) = V(f;a,x) (= \sup_{\pi \in \Pi(a,x)} V(f,\pi))$

Theorem 4.2. *$f \in BV(a,b) \Rightarrow V_f$ is an increasing function.*

Proof. Let $x, y \in (a,b)$ and $x < y$. $V_f(y) - V_f(x) = V(f;a,y) - V(f;a,x) = V(f;x,y) \geq 0$ □

Theorem 4.3. *$f \in BV(a,b) \Rightarrow V_f - f$ is an increasing function.*

Proof. $V_f(y) - f(y) - V_f(x) + f(x) = V(f;x,y) - (f(y) - f(x))$
Let $\pi = \{x, y\} \in \Pi(x,y)$. Since $V(f;x,y)$ is a supremum of $V(f,\pi)$, $V(f;x,y) - V(f,\pi) \geq 0$ □

Theorem 4.4. *$f \in BV(a,b) \Longleftrightarrow \exists g, h : [a,b] \longrightarrow \mathbb{R}$ increasing such that $f = g - h$*
$\Longrightarrow g = V_f, h = V_f - f$

Proof. $(\Longleftarrow) V(f,\pi) = V(g-h,\pi) \le V(g,\pi) + V(h,\pi)$ □

5 Continuous Functions of Bounded Variation

Theorem 5.1. *$f \in BV(a,b) \Longrightarrow \exists f(x^+) \forall x \in [a,b), \exists f(x^-) \forall x \in (a,b]$*

Proof. $\exists g, h$ such that $f = g - h$, g, h both increasing and have limit. □

Theorem 5.2. *$f \in BV(a,b), c \in [a,b]$*
f is continuous at $c \Longleftrightarrow V_f$ is also continuous at c.

Proof. $(\Longleftarrow)$ Let $\varepsilon > 0. \exists \delta > 0$ such that $|V_f(x) - V_f(c)| < \varepsilon$ whenever $|x - c| < \delta$.
Then $|f(x) - f(c)| \le |V_f(x) - V_f(c)| < \varepsilon$ whenever $|x - c| < \delta$.

$(\Longrightarrow)$ Let $\varepsilon > 0. \exists \delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$.
$\exists \pi \in \Pi(c,b)$ such that $V(f;c,b) - \varepsilon < V(f;\pi) \le V(f;c,b) = V(f;c,x_1) + V(f;x_1,b)$
We may assume $||\pi|| < \delta$. Then $V(f;c,b) - \varepsilon < |f(x_1) - f(c)| + \sum_2^p |f(x_j) - f(x_{j-1})| \le \varepsilon + V(f;x_1,b)$
$V(f;c,b) - V(f;x_1,b) \le 2\varepsilon$
$V(f;c,x_1) \le 2\varepsilon$
$V_f(x_1) - V_f(c) \le 2\varepsilon$ whenever $|x_1 - c| < \delta$.
Similarly to LHS. □

Theorem 5.3 (Corollary 3). *$f \in ([a,b] \cap BV(a,b)) \Longleftrightarrow V_f$ and $V_f - f$ monotone and increasing continuous.*