MAS241 CH4: Differentiation

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1 The Derivative

Definition 1.1. $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}, c \in I$. The derivative of f at c is the number $m = \lim_{k\to 0} \frac{f(c+h)-f(c)}{h}$ if it exists.

If it exists, we call f is differentiable at c and m is the derivative, $f'(c)$

Definition 1.2.

Theorem 1.1. *The line with slope m is tangent of f if and only if f is differentiable and m is the derivative.*

Definition 1.3. If f is differentiable at c, the mapping $t \longrightarrow mt = f'(c)t$ is called differential of f at c and denote it by df

Theorem 1.2. *Suppose f is differentiable at c, then*
1. $t \longrightarrow f'(c)t$ is linear
2. $^\forall \varepsilon > 0$ $^\exists N(0)$ such that $|f(c+h) - f(c) - mt| < \varepsilon|t|$ forall $t \in N(0)$

Theorem 1.3. *f is differentiable at c then f is continuous at c*

Theorem 1.4. *Algebra of derivatives*

2 Chain Rule

Theorem 2.1. $(f \circ g)'(c) = f'(g(c))g'(c)$

3 The Mean Value Theorem

Theorem 3.1. *Suppose that f has local maximum/minimum at $c \in (a, b)$ then $f'(c) = 0$*

Proof. The derivative of left side of c should be less than 0, right side of c should be larger than 0, so $f'(c) = 0$ □

Definition 3.1. If $f'(c) = 0$, c is called a critical point.

Theorem 3.2 (Roll's Theorem)**.** $f(a) = f(b) \implies {}^{\exists}c \in (a, b)$ *such that* $f'(c) = 0$

Proof. Prove using f is bdd above and below. find local max/min and compare to the edges. □

Theorem 3.3 (Mean Value Theorem)**.** $f : [a, b] \longrightarrow \mathbb{R}$, *continuous and differentiable,* $\implies^{\exists} c \in (a, b)$ *such that* $f'(c) = \frac{f(b)-f(a)}{b-a}$

Proof. Prove using Roll's theorem. □

Theorem 3.4 (Corollary 4)**.** $f : [a, b] \longrightarrow \mathbb{R}$, *continuous and differentiable.*
$f'(x) = 0$ *on* $[a, b]$*. Then,* f *is constant.*

Proof. Pick arbitrary x,y and use MVT. □

Theorem 3.5 (Corollary 5)**.** $f : [a, b] \longrightarrow \mathbb{R}$, *continuous and differentiable.*
If $f'(x) = g'(x)$*, then* $f(x) = g(x) + C$

Proof. Prove using corollary 4. □

Theorem 3.6 (Corollary 6)**.** $f : [a, b] \longrightarrow \mathbb{R}$, *continuous and differentiable*
$f'(x) > 0$*:* f *is strictly increases.*
$f'(x) < 0$*:* f *is strictly decreases.*

Proof. Pick arbitrary x,y and calculate slope. □

Theorem 3.7 (Cauchy's Generalized MVT)**.** $f, g : [a, b] \longrightarrow \mathbb{R}$, *continuous and differentiable.*
Then ${}^{\exists}c \in (a, b)$ *such that* $f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$

4 L'Hopital's Rule

Theorem 4.1. *$f, g[a,b] \longrightarrow \mathbb{R}$ is continuous and diff'ble, for $c \in (a,b)$, $f(c) = g(c) = 0$ and $g'(x) \neq 0$ on a neighborhood $N'(c, \delta)$, Then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$ if the right has a limit.*

Proof. Consider the right side limit.
By Cauchy's MVT, we have $f'(d)(g(x) - g(c)) = g'(d)(f(x) - f(c))$ for $c < d < x$.
$\frac{f'(d)}{g'(d)} = \frac{f(x)}{g(x)}$
$\lim_{x\to c+} \frac{f(x)}{g(x)} = \frac{f'(d)}{g'(d)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$
Because d is function of x and f', g' is continuous. □

Theorem 4.2. *$f, g(a, \infty) \longrightarrow \mathbb{R}$ is cont. and diff'ble, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, $g'(x) \neq 0$ $\Longrightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ if the right has a limit.*

Proof. Let $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$, Let $\varepsilon > 0$ be given.
Then $\exists \mu$ such that $|\frac{f'(x)}{g'(x)} - L| < \varepsilon$ whenever $x > M_1$.

Since $f(x) \longrightarrow \infty, g(x) \longrightarrow \infty, \exists M_2$ such that $f(x) > f(M_1), g(x) > g(M_1)$ for all $x > M_2$.
Let $x > M_2$. Then $\exists c_x \in (M_1, x)$ such that $f'(c_x)(g(x) - g(M_1)) = g'(c_x)(f(x) - f(M_1))$

$$\frac{f'(c_x)}{g'(c_x)} = \frac{f(x) - f(M_1)}{g(x) - g(M_1)} = \frac{f(x)(1 - f(M_1)/f(x))}{g(x)(1 - g(M_1)/g(x))} = \frac{f(x)}{g(x)}h(x)$$

$h(x) \longrightarrow 1$ as $x \longrightarrow \infty$, Hence $|h(x) - 1| < \varepsilon$ if x is large.
$\Longrightarrow |f(x)/g(x) - L| = |f'(x)/g'(x)h - L| = |f'(x)/g'(x)h - Lh + Lh - L|$
$= |f'(x)/g'(x) - L||h| + L|h - 1| \leq \varepsilon(1 + \varepsilon + L)$ □

5 Taylor's Theorem

Theorem 5.1. *$f, g[a, b] \longrightarrow \mathbb{R}$ is continuous and diff'ble up to order k+1.*
Let $x, x_0 \in [a, b]$ with $x \neq x_0$, p_k, q_k are Taylor's polynomial of f and g, respectively.
Then $\exists c \in (x, x_0)$ such that $f^{(k+1)}(c)[g(x) - q_k(x)] = g^{(k+1)}(c)[f(x) - p_k(x)]$

Proof. Let $a \leq x_0 < x \leq b$. Let $t \in [x_0, x]$.
Define $F(t) = f(t) + \sum f^{(j)}(t)/j!(x - t)^j$ (Taylor's polynomial centered at T) and $G(t)$ similarly.
Using Cauchy's MVT, $F'(c)(G(x) - G(x_0)) = G'(c)(F(x) - F(x_0))$.
$F(x_0) = p_k(x)$, $G(x_0) = q_k(x)$, $F(x) = f(x)$, $G(x) = g(x)$. $F'(t) = f^{(k+1)}(t)/k!(x - t)^k$.
And we can get the proposition. □

Theorem 5.2. *$f, g[a, b] \longrightarrow \mathbb{R}$ is continuous and diff'ble up to order k+1.*
Let $x, x_0 \in [a, b]$ with $x \neq x_0$. Then $\exists c \in (x_0, x)$ such that $f(x) = \sum f^{(j)}(x)/j!(x - x_0)^j + R(x_0; x)$, $R = f^{(k+1)}(c)/(k + 1)!(x - x_0)^{k+1}$.

Proof. Take $g(x) = (x - x_0)^{k+1}$ and apply thm1. □

Theorem 5.3. *$f : [a, b] \longrightarrow \mathbb{R}$ is diff'ble for all orders.*
Suppose $\exists M > 0$ such that $||f^{(k)}||_\infty \leq M^k$ for all k. Then $\sum f^{(j)}(x)/j!(x - x_0)^j \longrightarrow f$ uniformly on $[a, b]$.

Proof. $||f - p_k||_\infty$ goes to 0 as $k \longrightarrow \infty$

□