MAS241 CH3: Continuity

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1 Limit and Continuity

Definition 1.1. $f : S \subset \mathbb{R} \longrightarrow \mathbb{R}$, $c \in \overline{S}$

$\lim_{x\to c} f(x) = L \Longleftrightarrow^{\forall} \varepsilon > 0,^{\exists} \delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$ and $x \in S$.

Definition 1.2. $f : S \subset \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $c \in \overline{S}$

$\lim_{\mathbf{x}\to c} f(\mathbf{x}) = L \Longleftrightarrow^{\forall} \varepsilon > 0,^{\exists} \delta > 0$ such that $f(S \cap N'(c, \delta)) \subset (L - \varepsilon, L + \varepsilon)$ for $n = 1$.

Definition 1.3. $f : S \subset \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $c \in S$

f is called continuous at c if $\lim_{x\to c} f(x) = L = f(c)$.

Theorem 1.1. $f : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, $c \in \overline{S}$

$\lim_{x\to c} f(x) = L \Longrightarrow f$ is locally bounded at $x = c$.
Locally bounded means $^{\forall} nbd\ N(c)\ f$ is bounded.

Proof. Let $\varepsilon = 1$ then $^{\exists}\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$

$\Longrightarrow |f(x)| < |L| + 1$ whenever $x \in N'(c, \delta)$
$|f(x)| < |L| + 1 + |f(c)|$ whenever $x \in N(c, \delta)$ □

Theorem 1.2. $f : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, $c \in \overline{S}$

$\lim_{x\to c} f(x) = L > 0 \Longrightarrow f$ is locally bounded away from 0.

Proof. Since $f(x) \longrightarrow L > 0$ as $x \longrightarrow c$, there exists $\delta > 0$ such that $|f(x) - L| < L/2$ whenever $x \in N'(c, \delta)$.

Therefore, $f(x) > L/2$ □

Theorem 1.3. *Suppose that f_1 and f_2 are two real valued functions with common domain S in $\mathbb{R}^n$, that c is a point of $\overline{S}$, and that $\lim_{x\to c} f_1(x) = L_1$ and $\lim_{x\to c} f_2(x) = L_2$ exists. Then*

i) $\lim_{x\to c}[f_1(x) + f_2(x)] = L_1 + L_2$.
ii) For any constant a in $\mathbb{R}$, $\lim_{x\to c} a f_1(x) = aL_1$.
iii) $\lim_{x\to c} f_1(x) f_2(x) = L_1 L_2$.
iv) $\lim_{x\to c} 1/f_2(x) = 1/L_2$, provided that $L_2 \neq 0$.
v) $\lim_{x\to c} f_1(x)/f_2(x) = L_1/L_2$ provided that $L_2 \neq 0$.

Theorem 1.4 (Squeeze play theorem). *$f(x) \le g(x) \le h(x)$, $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$,*

Then $\lim_{x\to c} g(x) = L$

Theorem 1.5. *$f(x) \le g(x)$ in $N'(c, \delta) \cap S$*

$\lim_{x\to c} f(x) = L_1, \lim_{x\to c} g(x) = L_2 \implies L_1 \le L_2$

Characterizations of Discontinuities in $\mathbb{R}$

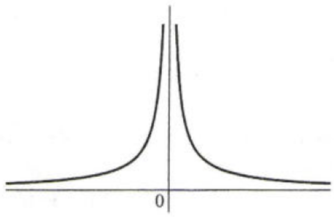

Figure 3.10 $f(x) = x^{-2}$ has a pole at $c = 0$

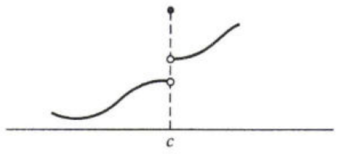

Figure 3.11 f has a jump discontinuity at $x = c$

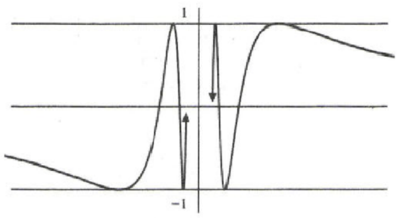

Figure 3.12 $f(x) = \sin\frac{1}{x}$ has an oscillatory discontinuity at $c = 0$

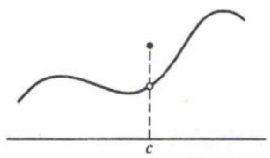

Figure 3.13 f has a removable discontinuity at $x = c$

2 Topological description of continuity

Theorem 2.1. *$f : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $T = f(S)$ We start with $m = 1$ case.*
f is continuous if and only if $f^{-1}(U)$ is relatively open in S for all U relatively open in T

Proof. ($\Longrightarrow$)
Let $\mathbf{x}_0 \in f^{-1}(U)$. Then $f(\mathbf{x}_0) \in U$.
Then $^\exists N(f(\mathbf{x}_0), \varepsilon) \subset U$. Since f is continuous, $^\exists N(\mathbf{x}_0, \delta) \cap S$ such that $f(N(\mathbf{x}, \delta) \cap S) \subset N(f(\mathbf{x}_0), \varepsilon)$
$N(\mathbf{x}, \delta) \cap S \subset f^{-1}(U)$ Therefore, all points in S are interior point, $f^{-1}(U)$ is open in S.

($\Longleftarrow$) Let $\varepsilon > 0$ be given, Let $\mathbf{x}_0 \in S$.
$f^{-1}(N(f(\mathbf{x}), \varepsilon))$ is open in S. Hence $^\exists N(\mathbf{x}_0, \delta) \subset f^{-1}(N(f(\mathbf{x}_0), \varepsilon))$
Hence f is continuous at $\mathbf{x}_0 \in S$. □

Theorem 2.2. *If S is a connected subset of $\mathbb{R}^n$ and if f is continuous on S,*
then $T = f(S)$ is also connected.

Proof. Suppose $f(S)$ is disconnected. Then $^\exists V, U$ open sets such that $V \cap U = \varnothing$, $f(S) \subset U \cup V$.
Then $f^{-1}(V), f^{-1}(U)$ are open. $f^{-1}(V) \cap f^{-1}(U) = \varnothing$
$S \subset f^{-1}(U) \cup f^{-1}(V), S \cap f^{-1}(V) \ne \varnothing, S \cap f^{-1}(U) \ne \varnothing$. □

Theorem 2.3. *If S is a compact subset of $\mathbb{R}^n$ and if f is continuous on S,*
then $T = f(S)$ is also compact.

Proof. "$f(S)$ is compact." Let $\{U_\alpha : \alpha \in A\}$ be an open cover of $f(S)$. Then $\{f^{-1}(U_\alpha)\}$ is an open cover of set S. Since S is compact, there exists a finite subcover $\{U_i : i = 1...N\}$.

Claim: $\{U_i : i = 1...N\}$ is a cover of $f(S)$.
$f(\mathbf{x}) \in f(S).f^{-1}(U_i)$ such that $\mathbf{x} \in f^-1(U_i)$. Then $f(\mathbf{x}) \in U_i$. Hence U_i is a finite open cover of $f(S)$.
Hence $f(S)$ is compact. □

Theorem 2.4. *If S is a compact subset of $\mathbb{R}^n$ and if f is continuous on S, then $f(x)$ has its max, min in S.*

Theorem 2.5 (Intermediate Value Theorem). *Let $f : [a, b] \longrightarrow \mathbb{R}$ and continuous*
1. $f(a)f(b) < 0 \Longrightarrow^\exists c \in (a, b)$ such that $f(c) = 0$
2. If f_0 is between $f(a)$ and $f(b)$, then $^\exists c \in (a, b)$ such that $f(c) = f_0$

Proof.
1. $[a, b]$ is connected. Hence $f([a, b])$ is connected.
Since $f([a, b])$ is an interval including 0, $^\exists c \in (a, b)$ such that $f(x) = 0$
2. Define $g([a, b]) \longrightarrow \mathbb{R}$ such that $g(x) = f(x) - f_0$.
Then g is continuous, $g(a)g(b) < 0$.
By 1, $^\exists c$ such that $g(c) = 0, f(c) = f_0$ □

Theorem 2.6 (The Generalized Intermediate Value Theorem). *Compact, connected set has a intermediate value.*

Theorem 2.7. *f is continuous at $\mathbf{x}_0$, g is continuous at $f(\mathbf{x}_0)$ then $g(f)$ is also continuous at $\mathbf{x}_0$*

Proof. Let $\varepsilon > 0$ be given. Then $|g(f(\mathbf{x})) - g(f(\mathbf{x}_0))| < \varepsilon$ with $|f(\mathbf{x}) - f(\mathbf{x}_0)| < \delta$ whenever $|\mathbf{x} - \mathbf{x}_0| < \delta$. And it is continuous. □

Limiting at ∞

Definition 2.1. $\lim_{||\mathbf{x}|| \to \infty} f(\mathbf{x}) \Longleftrightarrow^\forall \varepsilon > 0 \,^\exists M$ such that $|f(\mathbf{x}) - L| < \varepsilon$ whenever $||x|| > M$

3 Algebra of Continuous Function

$C(S)$ is the set of all functions continuous in S.

Theorem 3.1. *$S \subset \mathbb{R}^n, f, f_1, f_2 \in C(S)$*
$f_1 + f_2, af, f_1 f_2 \in C(S)$
$1/f, f_1/f_2 \in C(S)$ when $f, f_2 \neq 0$

Theorem 3.2. *f is continuous at $x_0 \in S \Longrightarrow f$ is locally bounded at x_0.*

$C_\infty(S)$ is the set of all functions bounded and continuous in S . If $f \in C_\infty(S) \Longrightarrow ||f||_\infty = \sup |f|$
If S is compact, $C_\infty(S) = C(S)$

4 Uniform Continuity

Definition 4.1. $f : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is called uniformly continuous if $^\forall \varepsilon > 0\ ^\exists \delta > 0$ such that $f(x) - f(y) < \varepsilon$ whenever $|x - y| < \delta$

Theorem 4.1.

Theorem 4.2. $f : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ *continuous and S is compact. Then f is uniformly continuous*

Proof. Let $\varepsilon > 0$ be given. The goal is to find $\delta > 0$ such that $|f(x)-f(y)| < \varepsilon$ whenever $|x-y| < \delta$
Then since f is continuous at $x\ ^\exists \delta(x)$ such that $|f(x) - f(y)| < \varepsilon/2$ for all $y \in N(x, \delta(x)|$
Then $\{N(x, \delta(x)/2) : x \in S\}$ open cover of S.
Since S is compact, we have a finite subcover, $\{N(x, \delta_i/2) : i = 1...N\}$

Take $\delta_0 = min(\delta_i/2)$
Let $|x - y| < \delta_0$. Then $^\exists x_i$ such that $x \in N(x_i, \delta_i/2)$
$|f(x) - f(y) \leq |f(x) - f(x_i)| + |f(x_i) - f(y)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$ whenever $|x - y| < \delta_0$ □

5 Uniform Norm, Uniform Convergence

Definition 5.1. $f \in C_\infty(S), ||f||_\infty = \sup |f(x)|$ in S.(supremum norm)

Theorem 5.1. *Supremum norm*($|| \cdot ||_\infty$ *satisfies conditions of the norm.*

Definition 5.2. $d_\infty(f \cdot g) = ||f - g||_\infty$ is the uniform metric.

Theorem 5.2. $d_\infty(f \cdot g)$ *is a metric.*

Definition 5.3. Neighborhood $N(f; \varepsilon) = \{g \in C_\infty(S) : ||g - f||_\infty < \varepsilon\}$

Definition 5.4. Uniform convergence
1. Let $F \subset C_\infty(S), f_0 \in C_\infty(S)$ is a uniform limit of F if $^\forall \varepsilon > 0,\ ^\exists f \in F$ such that $f \in N(f_0, \varepsilon)$
2. $\{f_k \in C_\infty(S)\}$ converges uniform to $f_0 \in C_\infty(S) \Longleftrightarrow\ ^\forall \varepsilon > 0\ ^\exists k_0$ such that $||f_k - f_0|| < \varepsilon$ whenever $k > k_0$
3. $\{f_k(x)\} \subset C_\infty(S)$ is called uniformly Cauchy if and only if $^\forall \varepsilon > 0\ ^\exists k_0$ such that $||f_k - f_m||_\infty < \varepsilon$ whenever $k, m > k_0$

[Pointwise convergence]
$f_k \longrightarrow f_0$ pointwise as $k \longrightarrow \infty$ if and only if for any $x \in S, f_k(x)$ converges to $f_0(x)$ as $k \longrightarrow \infty$

Theorem 5.3. $\{f_k\} \subset C_\infty(S), f_k \longrightarrow f_0$ *uniformly, then* $f_0 \in C_\infty(S)$ *(*$C_\infty(S)$ *is complete with uniform convergence)*

Proof. Let $x_0 \in S$. We will show f_0 is continuous at x_0.
Let $\varepsilon > 0$. Then $^\exists f_k \in N(f_0; \varepsilon/3)$
Since f_k is continuous, $^\exists \delta > 0$ such that $|f_k(x) - f_k(x_0)| < \varepsilon/3$ whenever $|x - x_0| < \delta$
$|f_0(x) - f_0(x_0)| \leq |f_0(x) - f_k(x)| + |f_k(x) - f_k(x_0)| + |f_k(x_0) - f_0(x_0)| \leq 3\varepsilon/3$
Hence f_0 is continuous.
$f_0 \in N(f_k, \varepsilon) \quad ||f_0||_\infty \leq ||f_k||_\infty + \varepsilon$ □

Theorem 5.4. *$\{f_k\} \subset C_\infty(S)$ is Cauchy sequence then there exists $M > 0$ such that $||f_k||_\infty < M \ {}^\forall k$*

Proof. Let $\{f_k\}$ be a Cauchy sequence. Then ${}^\forall\varepsilon > 0, {}^\exists k_0$ such that $||f_k - f_m||_\infty < \varepsilon$ whenever $k, m > k_0$
Take $\varepsilon = 1$, and $M = max(||f_{k_0}||_\infty + 1, ||f_i||_\infty)$
Then for $k \leq k_0$, theorem 4 holds.

Let $k > k_0$. Then $||f_k||_\infty = ||f_k - f_{k_0} + f_{k_0}||_\infty \leq ||f_k - f_{k_0}||_\infty + ||f_{k_0}||_\infty = \varepsilon + ||f_{k_0}||_\infty \leq M$ □

Theorem 5.5. *$C_\infty(S)$ is Cauchy complete.*

Proof. $\{f_k(x)\}$ is a Cauchy sequence for any $x \in S$. Then $f_k(x) \longrightarrow f_0(x)$ pointwise.
Claim: $f_k \in N(f_0, \varepsilon)\ {}^\forall k > k_0$
Suppose not. Then ${}^\exists k > k_0, x_0 \in S$ such that $|f_k(x_0) - f_0(x_0)| > \varepsilon$.
Since $f_m(x_0) \longrightarrow f_0(x_0)$, ${}^\exists m > k_0$ such that $|f_m(x_0) - f_k(x_0)| > \varepsilon$
Therefore, $||f_k - f_m||_\infty > \varepsilon$, which is contradiction.

Since the convergence is uniform, by theorem 3, f_0 is continuous. Furthermore, f_0 is bounded by theorem4. Hence $f_0 \in C_\infty(S)$ □

Theorem 5.6 (Corollary 6). *$C(S)$ is complete under uniform norm if S is compact.*

Proof. If S is compact, $C(S) = C_\infty(S)$ as it is bounded. □

Definition 5.5. $S \subset \mathbb{R}^n, F \subset C_\infty(S)$
F is dense in $C_\infty(S)$ in uniform norm if $N(f_0, \varepsilon) \cap F \neq \varnothing\ {}^\forall f_0 \in C_\infty(S)$

Theorem 5.7 (Weierstrass Approximation Theorem). *If $S \subset \mathbb{R}^n$ is compact, then collection of polynomials $P(S)$ is dense in $C_\infty(S)$ in uniform norm*

Bernstein's polynomial $B_k(x) = \sum_0^k f(j/k)\binom{k}{j} x^j(1-x)^{k-j}$

6 Vector valued functions

Definition 6.1. $f : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m, f(x) = (f^1(x), ..., f^m(x))$
f is continuous if f^i are all continuous.
$f(x) \longrightarrow V$ as $x \longrightarrow c\ (c \in \overline{S}) \Longleftrightarrow^\forall \varepsilon > 0\ {}^\exists\delta > 0$ such that $||f(x) - V|| < \varepsilon$ whenever $|x - c| < \delta$
f is continuous at $c \in S$ if $f(c) = V$

$N(f, \varepsilon) = \{g \in C_\infty(S) : ||f - g||_\infty < \varepsilon$
$||f||_\infty = \sup_{x \in S} ||f(x)||$