MAS241 CH2: Euclidean Space

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1 Euclidean space

For numbers in $\mathbb{R}$, $x_k \in \mathbb{R}$,$x_k \to x_0$ as $k \to \infty$
$\Leftrightarrow \forall \epsilon > 0 \exists k_0$ such that $\forall k > k_0, |x_k - x_0| < \epsilon$
For Euclidean space $\mathbb{R}^n$, $\mathbf{X} = (x^1, ...x^n) \in \mathbb{R}^n$
$\mathbf{X}_k \to \mathbf{X}_0$ as $k \to \infty$
$\Leftrightarrow \forall \epsilon > 0 \exists k_0$ such that $\forall k > k_0, ||\mathbf{x}_k - \mathbf{x}_0|| < \epsilon$

Definition 1.1. Inner product is defined $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x^i y^i$.

Theorem 1.1. $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
$\langle a\mathbf{x}, b\mathbf{y} \rangle = ab\langle \mathbf{x}, \mathbf{y} \rangle$

Definition 1.2. $||\mathbf{X}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \geq 0$
(Euclidean norm)

Theorem 1.2 (The Cauchy-Schwarz Inequality)**.** $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq ||\mathbf{x}|| \cdot ||\mathbf{y}||$

Proof. $\mathbf{z} = \mathbf{x} + t\mathbf{y}$, $t \in \mathbb{R}$.
$0 \leq ||\mathbf{z}||^2 = \langle \mathbf{z}, \mathbf{z} \rangle = \langle \mathbf{x} + t\mathbf{y}, \mathbf{x} + t\mathbf{y} \rangle = ||\mathbf{x}||^2 + 2t\langle \mathbf{x}, \mathbf{y} \rangle + t^2||\mathbf{y}||^2$
$\Rightarrow \langle \mathbf{x}, \mathbf{y} \rangle^2 - ||\mathbf{y}||^2||\mathbf{x}||^2 \leq 0$
And we call $\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}|| \cdot ||\mathbf{y}||}$ Cosine. □

Theorem 1.3. *For vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^n$*, the Euclidean norm has the following properties:*
i) Positive Definiteness: $||\mathbf{x}|| \geq 0$, $||\mathbf{x}|| = 0$ *when only* $\mathbf{x} = \mathbf{0}$.
ii) Absolute Homogeneity: $||c\mathbf{x}|| = |c| \cdot ||\mathbf{x}||$
iii) Subadditivity: $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$

Definition 1.3. If function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies i,ii,iii,
Then f is also called a norm.

Definition 1.4. Function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called a metric if
i) $d(\mathbf{x}, \mathbf{y}) \geq 0$ and $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.(Positivitiy)
ii)$d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ (Symmetry)
iii)$d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$ $\forall \mathbf{z}$ (Triangle inequality)

Definition 1.5. $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$

Theorem 1.4. *Euclidean metric is really a metric.*

Definition 1.6. $\mathbf{x}, \mathbf{y}$ are orthogonal to each other if $\cos\theta = 0$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$

Definition 1.7. $N(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{y}|| < r\}$
Deleted neighborhood is defined the minus of $\{\mathbf{x}\}$

Definition 1.8. $\varnothing \neq S \subset \mathbb{R}^n$, $\mathbf{x}_0 \in \mathbb{R}^n$ is called a limit point of S if and only if $\forall \epsilon > 0$ $\exists \mathbf{x} \in S$ such that $\mathbf{x} \in S \cap N'(\mathbf{x}_0, \epsilon)$

Definition 1.9. $S \subset \mathbb{R}^n$ is bounded if $\exists M > 0$ such that $||\mathbf{x}|| < M \ \forall \mathbf{x} \in S$

Sequences in $\mathbb{R}^n$

Definition 1.10. $\mathbf{x}_k \in \mathbb{R}^n$, $\mathbf{c} \in \mathbb{R}^n$,
$\mathbf{c}$ is a cluster point of $\{\mathbf{x}_k\}$ if and only if $\forall \epsilon > 0$, $\forall k_0 > 0 \exists k$ such that $||\mathbf{x}_k - \mathbf{c}|| < \epsilon$ whenever $k > k_0$

Definition 1.11. $\{\mathbf{x_k}\}$ converges to $\mathbf{x}_0 \in \mathbb{R}^n$ if $\forall \epsilon > 0 \exists k_0$ such that $||\mathbf{x}_k - \mathbf{x}_0|| < \epsilon$ whenever $k > k_0$ and $\mathbf{x}_0$ is called the limit point.

Definition 1.12. $\{\mathbf{x}_k\}$ is called Cauchy if and only if $\forall \epsilon > 0 \exists k_0$ such that $||\mathbf{x}_k - \mathbf{x}_m|| < \epsilon$ whenever $k, m > k_0$

Theorem 1.5.

Theorem 1.6.

Theorem 1.7. *Let* $\mathbf{x}_k = (x^1, ..., x^n) \in \mathbb{R}^n$, $\mathbf{x}_k$ *converges if and only if* x^k *converges.*

Theorem 1.8. $\mathbb{R}^n$ *is Cauchy Complete*

Theorem 1.9 (Generalized Bolzano-Weierstrass Theorem)**.** *Every bounded infinite set in* $\mathbb{R}^n$ *has a limit point in* $\mathbb{R}^n$

Proof. Split set to half and half again while having infinite elements.

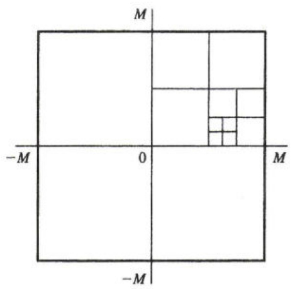

Figure 2.6

□

2 Open and Closed Set

Definition 2.1. Let S be any subset of $\mathbb{R}^n$ and let $\mathbf{x}$ be any point in $\mathbb{R}^n$.

(i) $\mathbf{x}$ is an interior point of S if $\exists r > 0$ such that $N(\mathbf{x}; r) \subseteq S$

(ii) $\mathbf{x}$ is a boundary point of S if $\forall r > 0 N(\mathbf{x}; r)$ contains both S and S^C

(iii) S is an open set if $\forall \mathbf{x} \in S$ is an interior point.(No boundary)

(iv) S is called a closed set if S contains every bdry points.

Theorem 2.1.
i) V_i is an open sets $\Rightarrow \bigcup_i V_i$ is open
ii) V_i is an open sets $\Rightarrow \bigcap_i^n V_i$ is open (n is finite)

Theorem 2.2. *Let set $C \subset \mathbb{R}^n$.*
C is closed $\Leftrightarrow C^c$ is open.

Theorem 2.3.
i) C_i is a closed sets $\Rightarrow \bigcap_i V_i$ is closed
ii) C_i is a closed sets $\Rightarrow \bigcup_i^n V_i$ is closed (n is finite)

Theorem 2.4. *Let set $C \subset \mathbb{R}^n$.*
C is closed $\Leftrightarrow C$ contains all of its limit points.

Definition 2.2. Let S be any subset of $\mathbb{R}^n$.

(i) The *interior* of S, denoted S^0, is the set of all interior points of S.

(ii) The *boundary* of S, denoted bd(S), is the set of all boundary points of S.

(iii) The *derived* of S, denoted S', is the set of all limit points of S.

(iv) The *closure* of S, denoted $\bar{S}$, is the union of S and S'.

(v) The *complement* of S, denoted S^c, is the set of all points in $\mathbb{R}^n$ not in S.

Theorem 2.5. *$S \subset \mathbb{R}^n$, S^0 is the union of all open sets contained in S.*
$S^0 = \bigcup_{V \in A} V, A = \{V | V \subset S, open\}$
Proof. We will prove it by using $A = B, A \subset B, B \subset A$

1. $(S^0 \subset \cup V)$ Let $\mathbf{x} \in S^0$, we can make $N(\mathbf{x}; r) < \epsilon$(which is an open subset)
2. $(\cup V \subset S^0)$ Let $\mathbf{x} \in \cup V$, Then $^\exists v \in A$ such that $\mathbf{x} \in V$
   Since V is an open set $^\exists r > 0$ such that $N(\mathbf{x}; r) \subset S^0$
   Hence $\mathbf{x}$ is an interior point, $\mathbf{x} \in S^0$

□

Theorem 2.6 (Corollary 6)**.** *S^0 is an open set.*

Theorem 2.7. $S \subset \mathbb{R}^n, \bar{S} = \bigcap_{V \in A} V, A = \{V | V \supset S, closed\}$

Theorem 2.8 (Corollary 8)**.** *$\bar{S}$ is closed.*

Definition 2.3. $\varnothing \neq S \subset \mathbb{R}^n, \mathbf{x} \in \mathbb{R}^n$
$d(S) = diam(S) = \sup\{||\mathbf{x} - \mathbf{y}|| / \mathbf{x}, \mathbf{y} \in S\}$

Definition 2.4. $dist(\mathbf{x}, S) = d(\mathbf{x}, S) = \inf\{||\mathbf{x} - \mathbf{y}|| / \mathbf{y} \in S\}$

Theorem 2.9. *Let S be any set in $\mathbb{R}^n$*

(i) $(S^0)^0 = S^0$

(ii) $(\bar{\bar{S}}) = \bar{S}$

(iii) $S^0 \cap bd(S) = \varnothing$

(iv) $S^0 \cup bd(S) = \bar{S}$

(v) $\bar{S} \cap \bar{S}^c = bd(S)$

Theorem 2.10. *Let S be a nonempty set in $\mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^n$*

(i) $d(\mathbf{x}, S) = 0 \Leftrightarrow \mathbf{x} \in \bar{S}$

(ii) *S is closed*$\Leftrightarrow^{\exists}$ $\mathbf{y} \in S$ *such that* $d(\mathbf{x}, S) = ||\mathbf{x} - \mathbf{y}||$

(iii) *S is closed*$\Leftrightarrow$ $dist(\mathbf{x}, S) > 0^{\forall}\mathbf{x} \notin S$

(iv) *S is open and* $\mathbf{x} \notin S \Leftrightarrow dist(\mathbf{x}, S) \neq ||\mathbf{x} - \mathbf{y}||^{\forall}\mathbf{y} \in S$

3 Completeness

Definition 3.1. A sequence $\{S_k\}$ of sets in $\mathbb{R}^n$ such that $S_k \supseteq S_{k+1}$ for each k in $\mathbb{N}$ is said to be *nested*

Theorem 3.1 (Cantor's Nested Interval Theorem)**.** *If* $\{I_k = [a_k, b_k]\}$ *is nested and* $\sup a_k = \alpha, \inf b_k = \beta$
$\bigcap I_k = [\alpha, \beta]$
Proof. $[\alpha, \beta] \subset [a_k, b_k] \forall k \Rightarrow [\alpha, \beta] \subset \bigcap I_k$
$\bigcap I_k \subset [\alpha, \beta]$: Let $x \notin [\alpha, \beta]$.
Then $x < \alpha or x > \beta$. Suppose $x < \alpha$, then $^{\exists}a_k$ such that $a_k < x < \alpha \Rightarrow x \notin I_k$.
β part is similar. □

Theorem 3.2 (Corollary 2)**.** *If* $\lim_{k\to\inf} b_k - a_k = 0$*, then* $\bigcap I_k$ *is a single point.*

Theorem 3.3 (Cantor's Criterion)**.** *If* $\{C_k\}$ *is a nested sequence of closed, bounded, nonempty subsets of* $\mathbb{R}^n$*, then*

$$\bigcap_{k=1}^{\infty} C_k \neq \varnothing$$

Furthermore, if $\lim_{k\to\infty} d(C_k) = 0$*, where* $d(C_k)$ *is the diameter of* C_k*,*
then $\bigcap_{k=1}^{\infty} C_k = \{\mathbf{x}_0\}$ *for some* $\mathbf{x}_0$ *in* $\mathbb{R}^n$
Proof. Since $C_k \neq \varnothing, ^{\exists}\mathbf{x}_k \in C_k\ ^{\forall}k$
$\{\mathbf{x}_k\}$ is bounded sequence.
Let C is a cluster point of $\{\mathbf{x}_k\}$
Claim: $C \in C_k^{\forall} k$
Since C_k is closed and $\mathbf{x}_j \in c_k\ ^{\forall}j \leq k$, $C \in C_k$
Therefore, $C \in \bigcap C_k$ □

Theorem 3.4. *Suppose that Cantor's criterion and Archimedes' principle hold in* $\mathbb{R}$*. Suppose also that* S *is a nonempty set in* $\mathbb{R}$ *that is bounded above. Then* $\sup S$ *exists in* $\mathbb{R}$*.*

Theorem 3.5. *The followings are equivalent*

1. *Axiom 1.1.1*
2. *Every bounded monotone sequence has a limit.*
3. $\mathbb{R}$ *has a Bolzano-Weierstrass property.*
4. $\mathbb{R}$ *is Cauchy-complete.*
5. $\mathbb{R}^n$ *is Cauchy-complete.*
6. $\mathbb{R}^n$ *has a Bolzano-Weierstrass property.*
7. *Cantor criterion is valid in* $\mathbb{R}^n$
8. *Cantor criterion is valid in* $\mathbb{R}$

4 Relative Topology and Connectedness

Topology: talking about open sets and closed sets.

Definition 4.1. $S \subset \mathbb{R}^n, S \subset X$ is relatively open(closed) in X if there exists an open(closed) set $U \subset \mathbb{R}^n$ such that $S = X \cap U$

Definition 4.2. $\varnothing \neq X \subset \mathbb{R}^n$

1. $N(\mathbf{x}_0; r) \cap X$ is (relative) neighborhood in X
2. $\{\mathbf{x}_k \in X\}$ converges in X if $^\exists \mathbf{x}_0 \in X$ such that $\mathbf{x}_n \to \mathbf{x}_0$
3. $S \subset X, \bar{S} \cap X$ is called relative closure

Theorem 4.1. $\varnothing \neq X \subset \mathbb{R}^n$*, The followings are equivalent:*

1. *X is Cauchy-complete.*
2. *X has Bolzano-Weierstrass property.*
3. *X satisfies Cantor's criterion.*

Proof. (i)⇒(ii):
Let $S \subset X$ be bounded an d has infinitely many points.

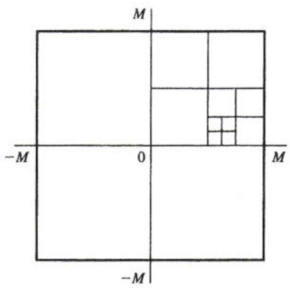

Figure 2.6

⇔ This sequence is Cauchy. (ii)⇒(iii):
Let $C_k \subset X$ be closed, bounded, nested non empty.
If one of C_k has only finite number of points, then the intersection is always not empty.
Suppose all of C_k contains infinite points, we can let $\mathbf{x}_k \in C_k$, $\mathbf{x}_k \neq \mathbf{x}_m, k \neq m$
$\{\mathbf{x}_k\}$ is bounded infinite points, then limit point should be $\mathbf{x}_0$ by B-W property. (iii)⇒(i):
... □

Definition 4.3. $X \in \mathbb{R}^n$ is called complete if one of the three properties of theorem 1 is satisfied

Theorem 4.2. $\varnothing \neq X \subset \mathbb{R}^n$ *is complete if and only if X is closed.*
Proof. (⇒) Let $\mathbf{x}_0$ be a boundary point of X. Then, we can make a Cauchy sequence that converges to $\mathbf{x}_0$. ⇒ $\mathbf{x}_0 \in X$, closed
(⇐) ... □

Definition 4.4. $S \subset \mathbb{R}^n$ is disconnected if
$^\exists U, V$ nonempty open sets such that $U \cap V = \varnothing, S \cap U \neq \varnothing, S \cap V \neq \varnothing, S \subset U \cup V$

Theorem 4.3. *Any interval $I = (a, b)$ is connected.*

5 Compactness

Definition 5.1. Let $\varnothing \neq S \subset \mathbb{R}^n$.
A collection of sets $C - \{U_\alpha \subset \mathbb{R}^n | \alpha \in A\}$ is called a *cover* of S if $\bigcup_{\alpha \in A} U_\alpha \supset S$. ($\alpha$ can be uncountable.)
If U_α are all open, then C is called an *open cover*.

Theorem 5.1 (Heine-Borel)**.** *Let S be any closed and bdd interval in $\mathbb{R}$ and let $\mathcal{C} = \{I_k\}$ be any open cover of S consisting open intervals. There exists a finite collection $\{I_{k_p}\}$ consisting of sets in $\mathcal{C}$ that also covers S.*

Definition 5.2. $S \subset \mathbb{R}^n$ is called *compact* if every open cover has a finite number of subcollections that covers S.

Theorem 5.2 (Lindelöf's Theorem)**.** *Let S be any subset of $\mathbb{R}^n$ and let $\mathcal{C} = \{U_\alpha : \alpha \text{ in } A\}$ be any open cover of S. Then some countable subcollection of $\mathcal{C}$ also covers S.*
Proof. Consider $B = \{N(\mathbf{y}, q) : \mathbf{y} \in \mathbb{Q}^n, q \in \mathbb{Q}\}$
$\mathbf{x} \in S$, Then $^\exists U_\alpha$ such that $\mathbf{x} \in U_\alpha$
Claim: $^\exists N(\mathbf{y}, q) \in B$ such that $\mathbf{x} \in N(\mathbf{y}, q) \subset U_\alpha$.
Let $r = dist(\mathbf{x}, U_\alpha^C)$. Consider $N(\mathbf{x}, r/3)$. Then there eexists $\mathbf{y} \in \mathbb{Q}^n$ such that it is in $N(\mathbf{x}, r/3)$
$^\exists q \in Q$ such that $r/3 < q < 2r/3$.
Let $A \subset B$ such that which consists of such $N(\mathbf{y}, q)$, it is countably many cover.
Consider a subcover of $\mathcal{C}$ that counts only one that include. □

Theorem 5.3 (The Generalized Heine-Borel Theorem)**.** *$S \subset \mathbb{R}^n$ is closed and bounded if and only if S is compact.*
Proof. Let $\{U_\alpha | \alpha \in A\}$ be an open cover. Then there exists a countable subcover by Lindelöf's Theorem that $\{U_i | i = 1...\infty\}$ be a countable cover.
Define $I_n = \bigcup_{i=1}^n U_i$, and $D_n = I_n^C \cap S$
Then D_k is closed, nested and bounded.
Claim: $\bigcap_{k=1}^\infty D_k = \varnothing$
Suppose not: Then $^\exists \mathbf{x} \in \bigcap^\infty D_k$, then $\mathbf{x} \in D_k \ ^\forall k$. $\mathbf{x} \in I_k^C \cap S \ ^\forall k$, and it is contradiction since U_i is a cover of S.
By Cantor's Criterion, $^\exists D_k = \varnothing \Rightarrow S \subset I_k$, and $U_i, i = 1...k$ is a finite open cover. □

Theorem 5.4.
(i) $S \subset \mathbb{R}^n$ is unbounded, then S is not compact.
(ii) $S \subset \mathbb{R}^n$ is not closed, then S is not compact.
Proof. Let $\mathbf{x}_0$ be a boundary point of S and $\mathbf{x}_0 \neq S$. Consider $\overline{N(\mathbf{x}_0, 1/k)}^C = U_k$.
Then $\bigcup_{k=1}^\infty U_k \supset S$ is an open cover but its finite union cannot cover S. □

Theorem 5.5. *$\varnothing \neq S \subset \mathbb{R}^n$, The followings are equivalent:*
(i) S is closed and bounded.
(ii) S is compact.
(iii) Every infinite subset of S has a limit point in S.